Computing holes in semi-groups and its applications to transportation problems

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Abstract

An integer feasibility problem is a fundamental problem in many areas, such as operations research, number theory, and statistics. To study a family of systems with no nonnegative integer solution, we focus on a commutative semigroup generated by a finite set of vectors in $\mathbb{Z}^d$ and its saturation. In this paper we present an algorithm to compute an explicit description for the set of holes which is the difference of a semi-group Q generated by the vectors and its saturation. We apply our procedure to compute an infinite family of holes for the semi-group of the $3 \times 4 \times 6$ transportation problem. Furthermore, we give an upper bound for the entries of the holes when the set of holes is finite. Finally, we present an algorithm to find all Q-minimal saturation points of Q.

1 Introduction

The linear integer feasibility problem is to ask whether the system

$$Ax = b, \quad x \geq 0, \tag{1}$$

where $A \in \mathbb{Z}^{d \times n}$ and $b \in \mathbb{Z}^d$, has an integral solution or not. In [11] we studied a *generalized integer feasibility problem*, that is, to find all b with no nonnegative integral solution for a given A. In recent years, the generalized integer linear feasibility problem has found applications in many research areas, such as number theory and statistics. For example, in number theory, the *Frobenius problem* is to find the largest positive integer b such that there does not exist an integral solution in (1) with $d = 1$ (e.g. [3], [2]). In statistics, one can find an application in the data security problem of *multi-way contingency tables* [9]. One of the challenge problems is the *3-dimensional integer planar transportation problem* (3-DIPTP), that is, the question to decide whether the set of *integer* feasible solutions of the $r \times s \times t$-transportation problem

$$\left\{ x \in \mathbb{Z}^{rst} : \sum_{i=1}^{r} x_{ijk} = u_{jk}, \sum_{j=1}^{s} x_{ijk} = v_{ik}, \sum_{k=1}^{t} x_{ijk} = w_{ij}, x_{ijk} \geq 0 \right\}$$

is empty or not for a given right-hand sides u, v, w. Vlach provides an excellent summary of attempts on 3-DIPTP [12]. For sequential importance sampling [8], non-existence of integral solution causes difficulties in its implementation.

Note that there exists a real nonnegative solution but there does not exist an integral nonnegative solution in (1) if and only if b is in the difference between the *semigroup* Q generated by the column vectors of A and its saturation $Q_{\text{sat}} = \text{cone}(A) \cap \text{lattice}(A)$, where $\text{cone}(A)$ is the cone generated by the columns of A and $\text{lattice}(A)$ is the lattice generated by the columns of A. We assume $\text{cone}(A)$ to be pointed. We call $H = Q_{\text{sat}} \setminus Q$ the set of *holes* of Q and call Q *normal* if $H = \emptyset$. H may be finite or infinite.

In this paper, we present an algorithm which gives a finite description of H. Practically, even with all the currently available software packages, checking normality of Q is still a difficult computational question. Computing a finite description of *all* elements in H is even more difficult. The reader should note that for fixed matrix sizes d and n, there exists a *polynomial size* encoding of the generating function $f(H; z) = \sum_{h \in H} z^h$ (where $z^h := z_1^{h_1} \cdots z_d^{h_d}$) as a rational generating function [11],[4]:

$$f(H; z) = \sum_{i \in I} \gamma_i \frac{z^{\alpha_i}}{\prod_{j=1}^{d} (1 - z^{\beta_{ij}})}.$$

Herein, I is a finite (polynomial size) index set and all the appearing data $\gamma_i \in \mathbb{Q}$ and $\alpha_i, \beta_{ij} \in \mathbb{Z}^d$ are of size polynomial in the input size of A. In fact, this observation is based on a result by Barvinok and Woods [5], who showed that there are such *short* rational function encodings for Q and for Q_{sat}, and consequently, also for $f(H; z) = f(Q_{\text{sat}}; z) - f(Q; z)$. Although the proof by Barvinok and Woods is constructive, its practical usefulness still has to be proven by an efficient implementation. In contrast to the *implicit* representation via rational generating functions, in this paper, we present an algorithm to compute an *explicit* representation of H, even for an infinite set H. Such an explicit representation need not be of polynomial size in the input size of A.

This paper is organized as follows: In Section 2 we set up basic notation and present our main results. Section 3 shows a combinatorial algorithm to compute the set of all *fundamental holes* of Q. In Section 4 we describe an algorithm to compute a finite representation of holes in Q. Section 5 shows an application of the algorithm to 3-DIPTP and in Section 6 we describe the bounds on the size of entries in each hole in Q. Finally in Section 7 we show an algorithm to compute the set of all *Q-minimal saturation points*.

2 Basic notation and the main results

The main result in this paper is the following.

Theorem 2.1. *There exists an algorithm that computes for an integral matrix A a finite explicit representation for the set H of holes of the semigroup Q generated by the columns of A, that is, the algorithm computes (finitely many) vectors $h_i \in \mathbb{Z}^n$ and monoids M_i, each given by a finite set of generators in $\mathbb{Z}^n$, $i \in I$, such that*

$$H = \bigcup_{i \in I} (\{h_i\} + M_i).$$

In fact, we explicitly present such an algorithm. Note that M_i could be trivial, that is, $M_i = \{0\}$. See Example 2.2 below for an example of such an explicit representation.

One basic object needed in our construction is the set F of fundamental holes. We call $h \in H$ *fundamental* if there is no other hole $h' \in H$ such that $h - h' \in Q$. Note that in contrast to H, F is always finite. For every hole $h \in H$ there exists a fundamental hole $f \in F$ such that $h \in f + Q$. In view of these facts our algorithm consists of the following two main steps:

1. First, compute the set F of fundamental holes.
2. Then, for each of the finitely many $f \in F$, compute an explicit representation of the holes in $f + Q$.

Moreover, we can use our algorithm to bound the entries of $h \in H$ in case that H is finite.

Theorem 6.1. *Let $A \subseteq \mathbb{Z}^{d\times n}$ be of full row-rank. Let $D(A)$ denote the maximum absolute value of the determinants of a $d \times d$ submatrix of A. Moreover, let $M_F(A) = \max_{i=1,\ldots,d} \sum_{j=1}^{n} |A_{ij}| - 1$. Then, if H is finite, the inequality*

$$\|h\|_\infty \leq (d+1) M_F^2(A) D(A)$$

holds for every $h \in H$.

Finally, we can use the above approach to compute the set of all Q-minimal saturation points of Q (Section 7). Herein, we call $s \in Q$ a *saturation point* of Q, if $s + Q_{\text{sat}} \subseteq Q$. The set of all saturation points of Q is denoted by S. We call $s \in S$ a Q-minimal saturation point if there is no other $s' \in S$ with $s - s' \in Q$. The set S of saturation points, considered as a Q-module, is often called a *conductor ideal* (e.g. [6]). S is finitely generated as a Q-module and hence the set of Q-minimal saturation points is finite.

We illustrate the above notions with the following simple example.

Example 2.2. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}$$

with the single fundamental hole $(1,1)^\intercal$ and with infinitely many holes

$$H = \{(1,1)^\intercal + \alpha \cdot (1,0)^\intercal : \alpha \in \mathbb{Z}_+\},$$

where $\mathbb{Z}_+$ denote the set of nonnegative integers. Q has three Q-minimal saturation points: $(1,2)^\intercal$, $(1,3)^\intercal$, and $(1,4)^\intercal$. See Figure 1. $\square$

In the following two sections we demonstrate how to perform the steps for Theorem 2.1 algorithmically. We accompany the theoretical construction with our running example, Example 2.2.

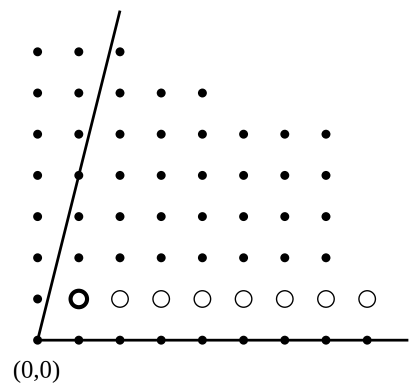

Figure 1: Non-holes, holes and fundamental hole for Example 2.2

3 Computing the fundamental holes F

In this section, we compute the set F of fundamental holes $h \in H$. To enumerate all fundamental holes, we first give a short proof that the number of fundamental holes is indeed finite. Let $A_{.i}$ denote the ith column of A.

Lemma 3.1 (Takemura and Yoshida [11]). *The set F of fundamental holes is a subset of*

$$P := \left\{ \sum_{i=1}^{n} \lambda_i A_{.i} : 0 \leq \lambda_1, \ldots, \lambda_n < 1 \right\} \cap \mathbb{Z}^d.$$

Proof. Each $f \in F$ lies in cone(A) and thus can be written as $f = A\lambda = \sum_{i=1}^{n} \lambda_i A_{.i}$ for some $\lambda \geq 0$. If $\lambda_j \geq 1$ for some $j \in \{1, \ldots, n\}$ then $f' = f - A_{.j} \in \text{cone}(A) \cap \text{lattice}(A)$ would contradict the Q-minimality of f, since $f - f' = A_{.j} \in Q$. Consequently, $\lambda_i < 1$ for all i. □

This shows that F is finite and also gives a finite procedure to enumerate F:

- Enumerate $P \cap \text{lattice}(A)$.
- Check for each $z \in P \cap \text{lattice}(A)$ whether z is a fundamental hole or not by solving $A\lambda = z, \lambda \in \mathbb{Z}_+^n$ and by checking whether $z - A_{.i} \in P \cap \text{lattice}(A) \subseteq Q$ for some i.

Practically, this construction can be sped-up as follows. First compute the (unique) minimal Hilbert basis (or better: integral basis) B of cone$(A)\cap$lattice(A). Again, similarly as above, one can show that $B \subseteq P$. If B contains no hole of Q, Q must be normal. Otherwise, every hole of Q appearing in B must be fundamental, since B is minimal. Finally, if $f \in F$ is not in B, f can be written as a nonnegative integer linear combination of elements in B, since $f \in \text{cone}(A) \cap \text{lattice}(A)$ and since B is a Hilbert basis (integral basis) of cone$(A) \cap$ lattice(A). This representation cannot have summands that are not fundamental holes, since otherwise f is would not be fundamental. To see this, let

$$f = \sum_{b \in B \cap F} \lambda_b b + \sum_{b \notin B \cap F} \mu_b b, \quad \lambda_b, \mu_b \in \mathbb{Z}_+ \; \forall b,$$

with $\sum_{b\not\in B\cap F}\mu_b b \neq 0$. Observe, that

$$f' = \sum_{b\in B\cap F} \lambda_b b$$

must be a hole of Q, as otherwise f is not a hole. But since

$$f - f' = \sum_{b\not\in B\cap F} \mu_b b \in Q,$$

f cannot be a fundamental hole.

Thus we can enumerate F as follows:

- Compute the Hilbert basis (integral basis) B of $\text{cone}(A) \cap \text{lattice}(A)$.
- Check whether each $z \in B$ is a fundamental hole or not, that is, compute $B \cap F$.
- Generate all nonnegative integer combinations of elements in $B \cap F$ that lie in P and check for each such z whether it is a fundamental hole or not.

Example 2.2 cont. In our example, the lattice L generate by the columns of A is simply $\text{lattice}(A) = \mathbb{Z}^2$. With this, the Hilbert basis B of $\text{cone}(A) \cap \text{lattice}(A)$ consists of 5 elements:

$$B = \{(1,0)^\intercal, (1,1)^\intercal, (1,2)^\intercal, (1,3)^\intercal, (1,4)^\intercal\},$$

out of which only $(1,1)^\intercal$ is a hole. Being in B, $(1,1)^\intercal$ must be a fundamental hole. Thus, $B \cap F = \{(1,1)^\intercal\}$. Constructing nonnegative integer linear combinations of elements from $B \cap F$, we already see that the combination $2 \cdot (1,1)^\intercal = (2,2)^\intercal$ is an element of Q and consequently, there is no other fundamental hole in Q, i.e. $F = \{(1,1)^\intercal\}$. □

4 Computing the holes in $f + Q$

In this section we discuss how to compute the holes in $f + Q$ for each fundamental hole $f \in F$. Note that a point $z \in f+Q$ is either a hole or it belongs to Q. That is, every non-hole in $f+Q$ belongs to $(f+Q)\cap Q$. Moreover, if $z \in (f+Q)\cap Q$ then also $z + A\lambda \in (f+Q)\cap Q$ for all $\lambda \in \mathbb{Z}_+^n$. Thus we define a monomial ideal $I_{A,f} \in \mathbb{Q}[x_1,\ldots,x_n]$ by

$$I_{A,f} = \langle x^\lambda : \lambda \in \mathbb{Z}_+^n, f + A\lambda \in (f+Q)\cap Q\rangle. \qquad (2)$$

By construction, $f+A\lambda$, $\lambda \in \mathbb{Z}_+^n$, is not a hole of Q if and only if $x^\lambda \in I_{A,f}$. Therefore, we are looking for an explicit description of the monomials *not* belonging to the monomial ideal $I_{A,f}$. These monomials are usually called *standard monomials* and there are algorithms to compute an explicit disjoint or non-disjoint representation of them once ideal generators for $I_{A,f}$ are known. Via the (typically non-injective) linear transformation $x^\lambda \mapsto f + A\lambda$, one recovers an explicit (usually non-disjoint) representation of all holes of Q in $f + Q$.

It remains to find (minimal) generators for $I_{A,f}$. The minimal generators correspond to the $\leq$-minimal elements in the set

$$L_{A,f} = \{\lambda \in \mathbb{Z}_+^n : \exists \mu \in \mathbb{Z}_+^n \text{ such that } f + A\lambda = A\mu\}.$$

To compute these minimal elements directly inside this projection is a hard computational task and deserves further investigation. Let us therefore compute a usually *non-minimal* generating set for $I_{A,f}$ from a higher-dimensional problem.

Lemma 4.1. *Let M be the set of $\leq$-minimal solutions (λ, μ) to $f+A\lambda = A\mu$, $(\lambda, \mu) \in \mathbb{Z}_+^{2n}$. Then*

$$I_{A,f} = \langle x^\lambda : \exists \mu \in \mathbb{Z}_+^n \textit{ such that } (\lambda, \mu) \in M\rangle.$$

Proof. Let $\lambda_0 \in L_{A,f}$ be $\leq$-minimal. We show now that there exists some $\mu_0 \in \mathbb{Z}_+^n$ such that (λ_0, μ_0) is a $\leq$-minimal solution to $f + A\lambda = A\mu$, $(\lambda, \mu) \in \mathbb{Z}_+^{2n}$. Then, as claimed, the minimal generator x^{λ_0} is contained in the given set of generators for $I_{A,f}$.

Suppose on the contrary, that for every $\mu \in \mathbb{Z}_+^n$ the vector (λ_0, μ) is *not* a $\leq$-minimal solution to $f + A\lambda = A\mu$, $(\lambda, \mu) \in \mathbb{Z}_+^{2n}$. Let μ_0 be a $\leq$-minimal solution to $f + A\lambda_0 = A\mu$, $\mu \in \mathbb{Z}_+^n$. Then, by our assumption, there is some vector $(\lambda', \mu') \in \mathbb{Z}_+^{2n}$ with $f + A\lambda' = A\mu'$, $(\lambda', \mu') \leq (\lambda_0, \mu_0)$, and $(\lambda', \mu') \neq (\lambda_0, \mu_0)$. If $\lambda' \neq \lambda_0$ holds, we have a contradiction to λ_0 being $\leq$-minimal in $L_{A,f}$. If $\lambda' = \lambda_0$ and $\mu' \neq \mu_0$ holds, we have a contradiction to μ_0 being a $\leq$-minimal solution to $f + A\lambda_0 = A\mu$, $\mu \in \mathbb{Z}_+^n$. This shows that (λ_0, μ_0) is a $\leq$-minimal solution to $f + A\lambda = A\mu$, $(\lambda, \mu) \in \mathbb{Z}_+^{2n}$, as we wanted to show. $\square$

Example 2.2 cont. Let $f = (1, 1)^\intercal$ and consider $(f + Q) \cap Q$. The linear system to solve is

$$\begin{array}{ccccccccccccccccc} 1 & + & \lambda_1 & + & \lambda_2 & + & \lambda_3 & + & \lambda_4 & = & \mu_1 & + & \mu_2 & + & \mu_3 & + & \mu_4 \\ 1 & & & + & 2\lambda_2 & + & 3\lambda_3 & + & 4\lambda_4 & = & & & 2\mu_2 & + & 3\mu_3 & + & 4\mu_4 \end{array}$$

with $\lambda_i, \mu_j \in \mathbb{Z}_+, i, j \in \{1, 2, 3, 4\}$.

`4ti2` gives the following 5 minimal inhomogeneous solutions $(\lambda, \mu) \in \mathbb{Z}_+^8$:

$(0, 0, 0, 2, 0, 0, 3, 0)^\intercal, (0, 1, 0, 0, 1, 0, 1, 0)^\intercal, (0, 0, 1, 0, 1, 0, 0, 1)^\intercal (0, 0, 1, 0, 0, 2, 0, 0)^\intercal, (0, 0, 0, 1, 0, 1, 1, 0)^\intercal.$

Thus, we get the monomial ideal

$$I_{A,f} = \langle x_4^2, x_2, x_3, x_3, x_4\rangle = \langle x_2, x_3, x_4\rangle,$$

whose set of standard monomials is $\{x_1^\alpha : \alpha \in \mathbb{Z}_+\}$. Thus, the set of holes in $f + Q$ is explicitly given by

$$\{f + \alpha A_{.1} : \alpha \in \mathbb{Z}_+\} = \{(1, 1)^\intercal + \alpha(1, 0)^\intercal : \alpha \in \mathbb{Z}_+\},$$

as already claimed above. $\square$

5 Infinitely many holes for $3 \times 4 \times 6$ transportation problem

In this section, we apply the procedure from the last section to the semi-group Q spanned by the matrix A defining a $3 \times 4 \times 6$ transportation problem. Already in 1986, Vlach [12] has shown that this semi-group is not normal by explicitly stating a hole f, which is fundamental. He showed that $Ax = f, x \geq 0$ has a (unique) rational solution, which in turn is fractional. In the following, we construct a finite representation of all holes of Q belonging to $f + Q$. We show that there are in fact *infinitely many* such holes.

The semi-group of the $3 \times 4 \times 6$ transportation problem is of special interest, as it is the smallest three-dimensional transportation problem for which it is known that the associated semi-group is not normal. The only two cases of (also higher-dimensional) transportation problems for which the normality question is still open are those of sizes $3 \times 4 \times 5$ and $3 \times 5 \times 5$ [10]. The previously open case $3 \times 4 \times 4$ has been solved by the third author using the software package NORMALIZ [7]. The associated semi-group is normal.

For the $3 \times 4 \times 6$ problem, a fundamental hole f given by Vlach [12] is defined by the following three matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The unique point in this $3 \times 4 \times 6$ transportation polytope $\{z \in \mathbb{R}^{72} : Az = f, z \geq 0\}$ is

$$z^* = \frac{1}{2} \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

If $A_{.,ijk}$ denotes the column of A corresponding to variable z_{ijk}, then the monomial ideal $I_{A,f}$ constructed in the previous section is generated by the 48 monomials x_{ijk} for which $z^*_{ijk} = 0$. This can be shown as follows.

Firstly, $1 \notin I_{A,f}$, since $f \notin Q$. Secondly, using `4ti2`, one verifies for each of these 48 indices that $f + A_{.,ijk} = A\mu$ has a nonnegative integer solution $\mu \in \mathbb{Z}^{72}$, by explicitly constructing such a solution. Finally, as it remains to look only for ideal generators of $I_{A,f}$ not divisible by the 48 monomials x_{ijk} for which $z^*_{ijk} = 0$, the linear system from the previous section simplifies to

$$f + A'\lambda' = A\mu, \lambda \in \mathbb{Z}^{24}_+, \mu \in \mathbb{Z}^{72}_+,$$

where A' is formed out of the 24 columns $A_{.,ijk}$ of A for which $z^*_{ijk} > 0$. This system does not have an integral solution. In fact, the only real solution is $(\lambda', \mu) = (0, z^*)$. To see this one either solves this system, for example using `4ti2`, or one observes that the vector $f + A'\lambda'$ has many zero entries that are present for arbitrary choices of λ'. These

zero entries imply that $\mu_{ijk} = 0$ for all triples ijk for which $z^*_{ijk} = 0$. The remaining linear system

$$f + A'\lambda' = A\mu', \lambda \in \mathbb{Z}_+^{24}, \mu' \in \mathbb{Z}_+^{24},$$

has a unique real solution, namely $(0, z^{*\prime})$, which can be checked by applying Gaussian elimination to the system $A(\mu' - \lambda') = f, (\mu' - \lambda') \in \mathbb{R}^{24}$.

Thus, the set of holes of Q belonging to $f + Q$ are given by $f + \text{semi-group}(A')$.

6 Computing bounds

For this section, let us assume that the set H is finite. We will now use our approach above to establish bounds on the size of the entries for each $h \in H$. Clearly, such a bound can then be used to show that H cannot be finite if a hole with sufficiently big entries has been found.

Theorem 6.1. *Let $A \subseteq \mathbb{Z}^{d\times n}$ be of full row-rank. Let $D(A)$ denote the maximum absolute value of the determinants of a $d \times d$ submatrix of A. Moreover, let $M_F(A) = \max_{i=1,\ldots,d} \sum_{j=1}^n |A_{ij}| - 1$. Then, if H is finite, the inequality*

$$\|h\|_\infty \leq (d+1)M_F^2(A)D(A)$$

holds for every $h \in H$.

Proof. First, we can bound the elements $f \in F$ using the relation

$$F \subseteq \left\{ \sum_{j=1}^n \lambda_j A_{.j} : 0 \leq \lambda_1, \ldots, \lambda_n < 1 \right\}.$$

Thus,

$$|f^{(i)}| \leq \sum_{j=1}^n |A_{ij}| - 1 \leq \max_{i=1,\ldots,d} \sum_{j=1}^n |A_{ij}| - 1 =: M_F(A)$$

holds for all $f \in F$ and all $i = 1, \ldots, d$.

Next, as H is finite, all ideals $I_{A,f}$, $f \in F$, must have a finite set of standard pairs, which is equivalent to saying that there must be a monomial generator $x_i^{\alpha_i}$ for every $i = 1, \ldots, n$. Such a monomial generator corresponds to a minimal inhomogeneous solution (α_i, μ) to $f + \alpha_i A_{.i} = A\mu, \alpha_i \in \mathbb{Z}_+, \mu \in \mathbb{Z}_+^n$. Let us now bound the values for such a minimal α_i.

First, the minimal inhomogeneous solutions (α_i, μ) to $f + \alpha_i A_{.i} = A\mu, \alpha_i \in \mathbb{Z}_+, \mu \in \mathbb{Z}_+^n$ correspond exactly to the minimal homogeneous solutions to $fu + \alpha_i A_{.i} - A\mu = 0, \alpha_i, u \in \mathbb{Z}_+, \mu \in \mathbb{Z}_+^n$ with $u = 1$. Each entry in a minimal homogeneous solutions, however, can be bounded by $(d+1)$ times the maximum absolute value $D(f\ A_{.i}A)$ of the determinants of a maximal submatrix of the defining matrix $(f\ A_{.i}A)$.

Thus, in particular,

$$\alpha_i \leq (d+1)D(f\ A_{.i}A) \leq (d+1) \max_{j=1,\ldots,d} |f^{(j)}| \cdot D(A_{.i}\ \ -A) = (d+1)M_F(A) \cdot D(A).$$

Consequently, we can bound the entries of a hole in $f+Q$ by giving bounds for

$$f+\sum_{j=1}^{n}(\alpha_j-1)A_{.j}.$$

For $h\in(f+Q)\cap H$, the ith entry is bounded as

$$\begin{aligned}
h^{(i)} &\leq |f^{(i)}|+\sum_{j=1}^{n}(\alpha_j-1)|A_{ij}| \\
&\leq M_F(A)+\sum_{j=1}^{n}((d+1)M_F(A)D(A)-1)|A_{ij}| \\
&= M_F(A)+((d+1)M_F(A)D(A)-1)\sum_{j=1}^{n}|A_{ij}| \\
&\leq M_F(A)+((d+1)M_F(A)D(A)-1)M_F(A) \\
&= (d+1)M_F^2(A)D(A).
\end{aligned}$$

As this bound is independent on $f\in F$, we have

$$\|h\|_\infty \leq (d+1)M_F^2(A)D(A) \quad \forall h\in H,$$

if H is finite. $\square$

Example 2.2 cont. In our example, we have

- $d+1=3$,
- $M_F(A)=\max(1+1+1+1,0+2+3+4)=9$, and
- $D(A)=\max|2\times 2 \text{ determinant of } A| = |\det\left(\begin{smallmatrix}1&1\\0&4\end{smallmatrix}\right)|=4$.

Thus, if H was finite, we would get the bound $\|h\|_\infty\leq 3\cdot 9^2\cdot 4=972$. In our example, however, one can easily verify that $(1000,1)$ is a hole. Moreover, it violates the computed bound. Consequently, H cannot be finite. $\square$

7 Computing all Q-minimal saturation points

In this section, let S denote the set of saturation points of Q, that is, the set of all those $s\in Q$ such that $s+Q_{\text{sat}}\subseteq Q$. Let us now show how the above approach can be used in order to compute $\min(S;Q)$, the set of all Q-minimal points in S. We also recover the known fact that $\min(S;Q)$ is always finite. We state the following theorem.

Theorem 7.1.

$$S = \bigcap_{f \in F} [((f+Q) \cap Q) - f]$$

and hence

$$S = \{A\lambda \mid x^\lambda \in \bigcap_{f \in F} I_{A,f}\},$$

where $I_{A,f}$ is defined in (2).

Proof.

$$\begin{aligned}
s \in S \quad &\Leftrightarrow \quad s \in Q \text{ and } s + Q_{\text{sat}} \subseteq Q \quad \text{(by definition)} \\
&\Leftrightarrow \quad s \in Q \text{ and } s + H \subseteq Q \quad \text{(since } Q_{\text{sat}} = Q \cup H \text{ and } s + Q \subseteq Q,\ \forall s \in Q) \\
&\Leftrightarrow \quad s \in Q \text{ and } s + F \subseteq Q \quad \text{(since } H \subseteq F + Q) \\
&\Leftrightarrow \quad s + f \in f + Q \text{ and } s + f \subseteq Q \quad \forall f \in F \\
&\Leftrightarrow \quad s + f \in (f+Q) \cap Q \quad \forall f \in F.
\end{aligned}$$

Consequently, we have $s \in S \Leftrightarrow s \in \bigcap_{f \in F}[((f+Q) \cap Q) - f]$. Furthermore with $s = A\lambda$ for some $\lambda \in \mathbb{Z}_+^n$ (as $s \in Q$), we get $s \in S \Leftrightarrow x^\lambda \in \bigcap_{f \in F} I_{A,f}$. □

Define

$$I_A = \bigcap_{f \in F} I_{A,f}.$$

Then I_A is a monomial ideal being the intersection of the monomial ideals $I_{A,f}$. I_A can be found algorithmically, for example with the help of Gröbner bases. The elements $s \in \min(S;Q)$ correspond exactly to the minimal ideal generators x^λ of I_A via the relation $s = A\lambda$. (Note, however, that this relation need not be one-to-one. There may be many minimal ideal generators corresponding to the same Q-minimal saturation point.)

Example 2.2 cont. In our example, we have $I_A = I_{A,f} = \langle x_2, x_3, x_4 \rangle$, as there exists only one fundamental hole f. The three generators of I_A correspond to the three Q-minimal saturation points $(1,2)^\intercal$, $(1,3)^\intercal$, and $(1,4)^\intercal$. □

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